Using Chessboards to investigate an Unsolved Conjecture in Graphs

So you want to learn about graphs?

Hanzhang Yin

University of Connecticut

July 15th, 2022

Graph

$$G = \{V(G), E(G)\}$$

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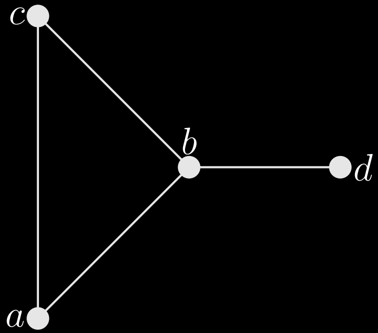
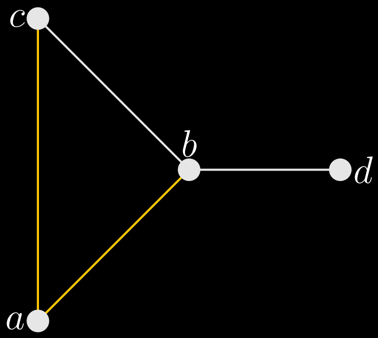
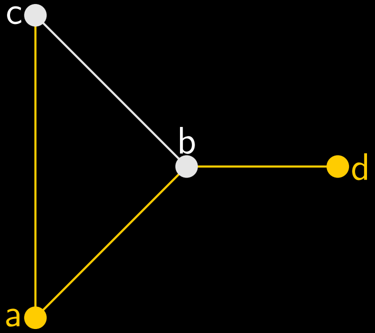


Adjacency



Dominating set

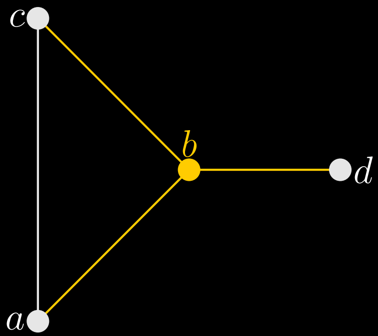


$\{a, d\}$ is a dominating set.

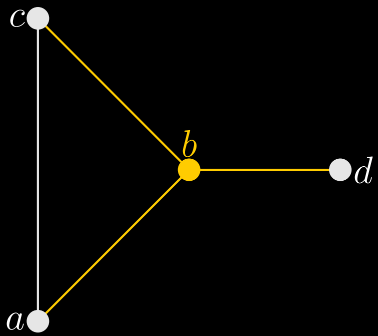
Dominating set

$\gamma(G)$: the number of vertices in a smallest dominating set of graph G.

Dominating set

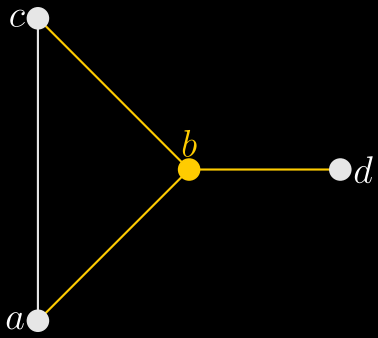


Dominating set



$\{b\}$ is a dominating set.

Dominating set



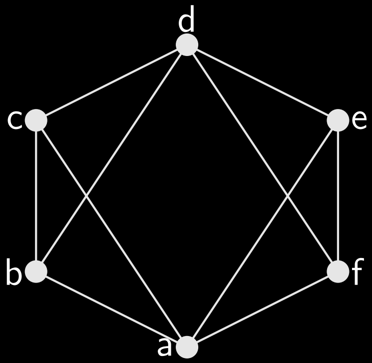
$\{b\}$ is a dominating set.

$\gamma(G) = 1.$

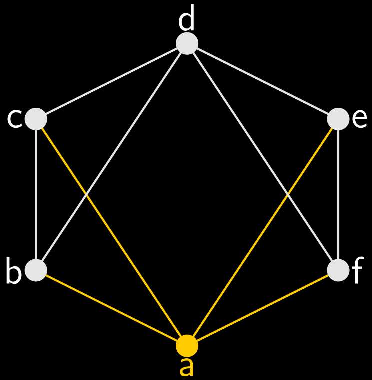
k-Dominating set

$\gamma_k(G)$: the number of vertices in a smallest k-dominating set of graph G.

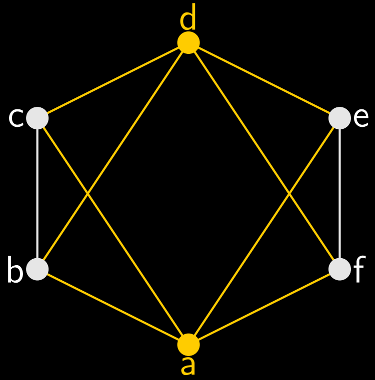
2-Dominating set



2-Dominating set



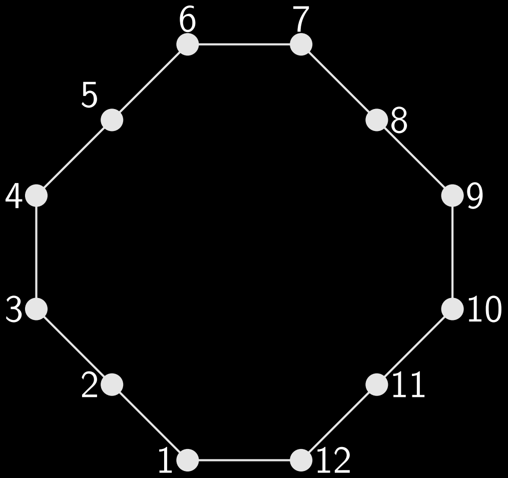
2-Dominating set



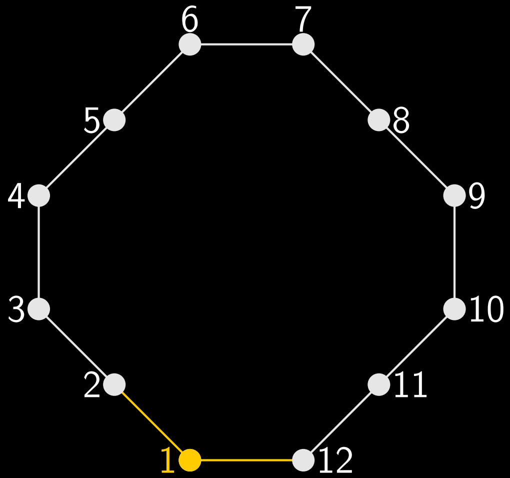
$\{a, d\}$ is a 2-dominating set.

$\gamma_2(G) = 2$.

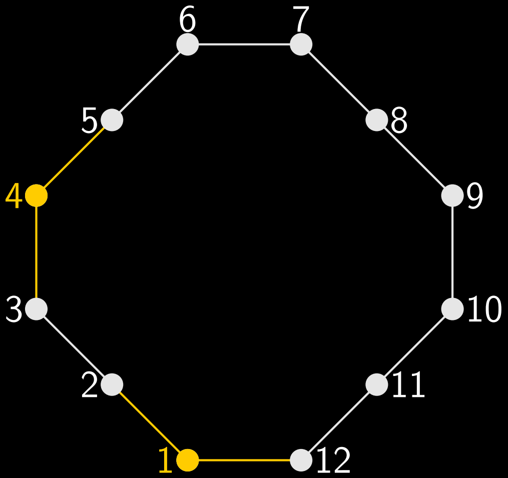
k-Dominating set



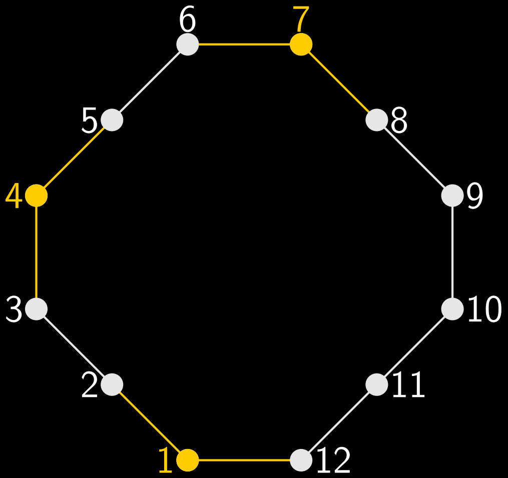
k-Dominating set



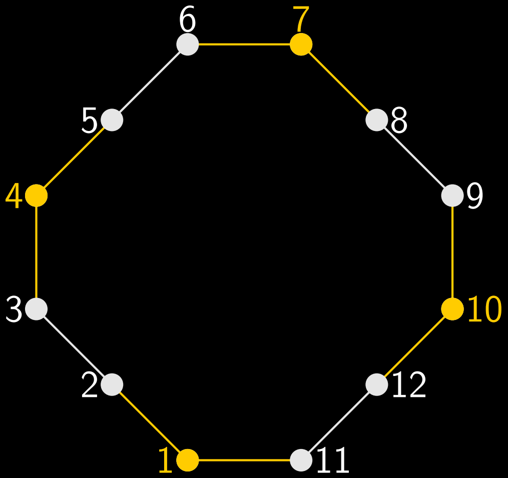
k-Dominating set



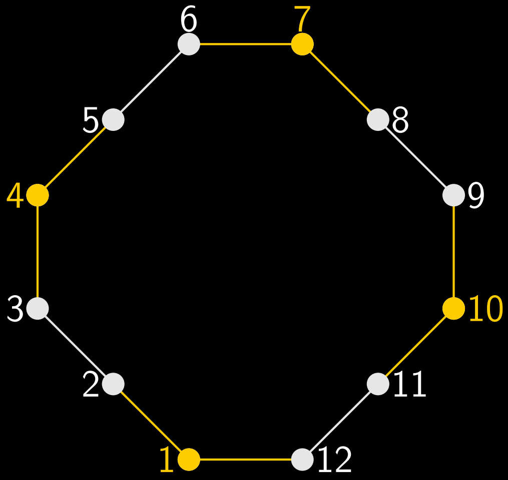
k-Dominating set



k-Dominating set

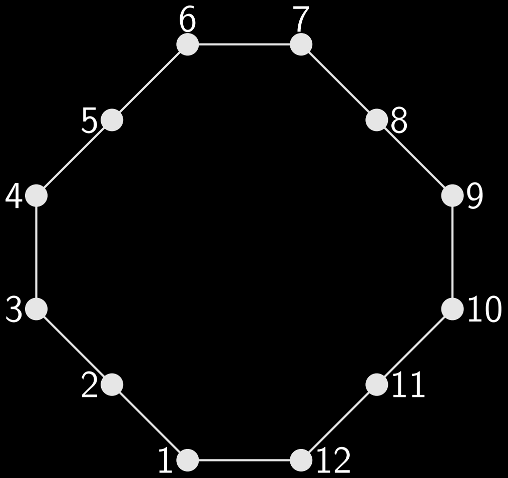


k-Dominating set

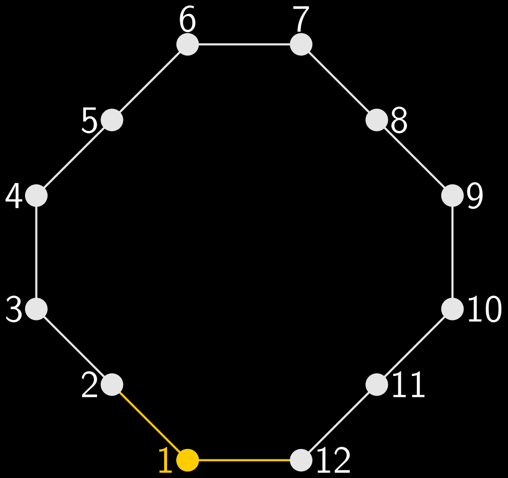


$\{1, 4, 7, 10\}$ is a 1-dominating set and $\gamma(G) = 4$

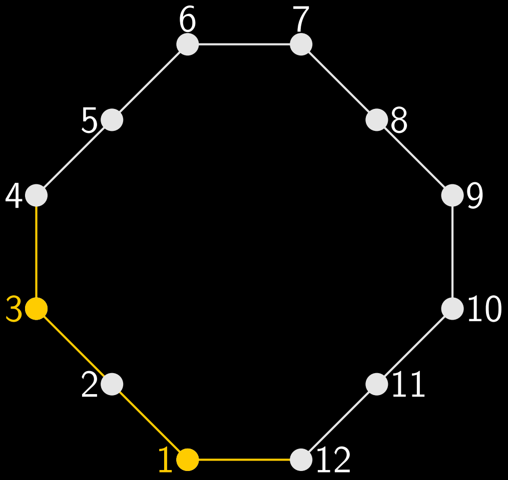
k-Dominating set



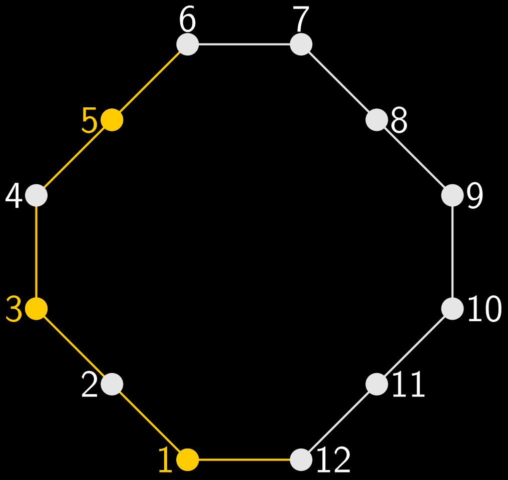
k-Dominating set



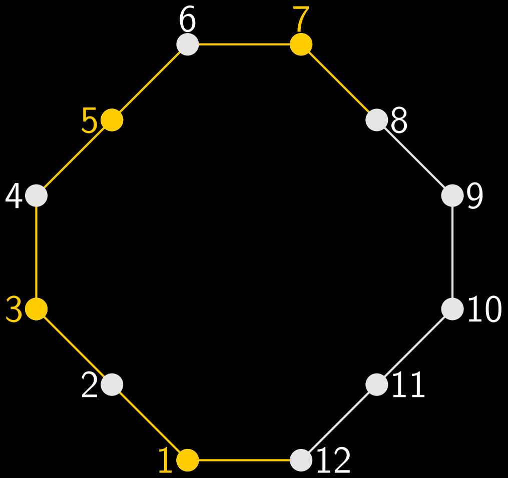
k-Dominating set



k-Dominating set



k-Dominating set



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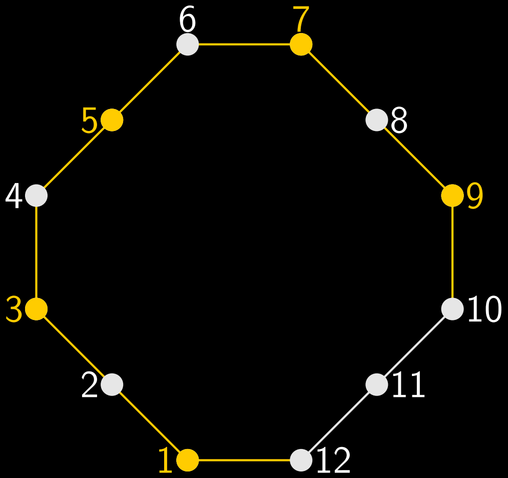

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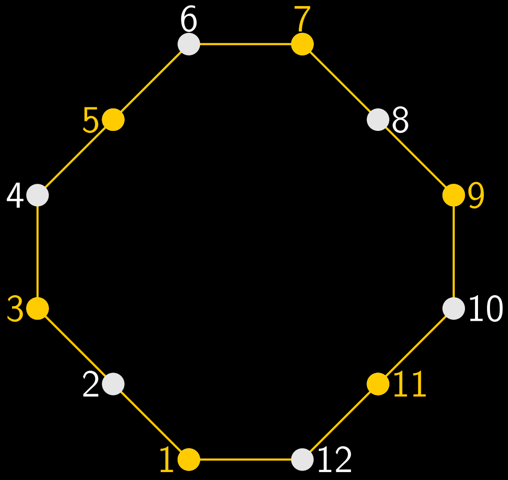
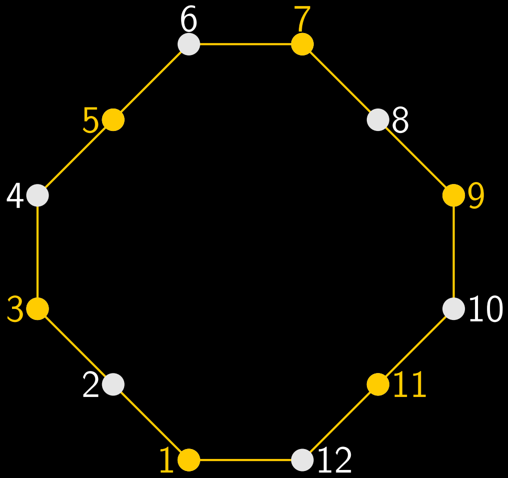


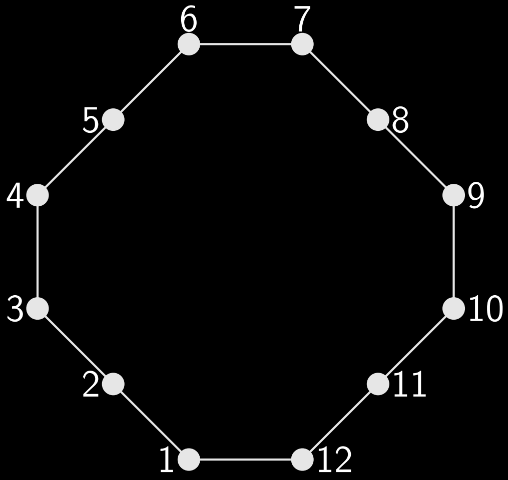
$\{1, 3, 5, 7, 9, 11\}$ is a 2-dominating set and $\gamma_2(G) = 6$.

k-Dominating set



$\{1, 3, 5, 7, 9, 11\}$ is a 1-dominating set.

k-Dominating set



$$\gamma(G) \leq \gamma_2(G)$$

K-Dominating set

For any graph G,

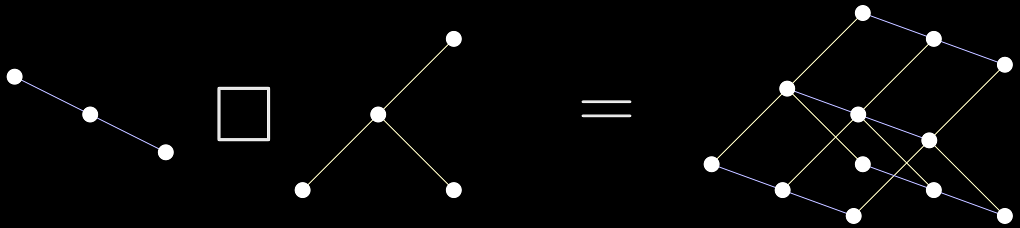
$$\gamma_k(G) \leq \gamma_{k+1}(G).$$

Cartesian Product of Graphs

Cartesian Product of Graphs

$$G \Box H = \{V(G \Box H), E(G \Box H)\}$$

Cartesian Product of two graphs

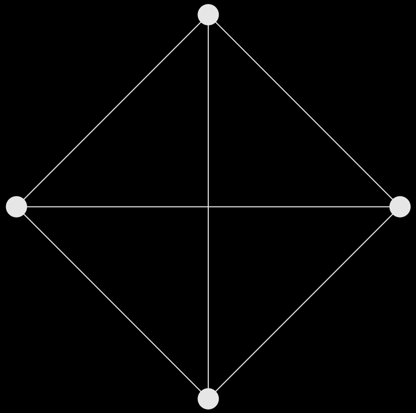


Complete Graph

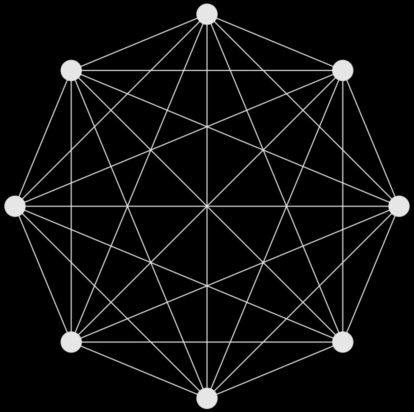
Complete Graph

$$K_n$$

K_4



K_8



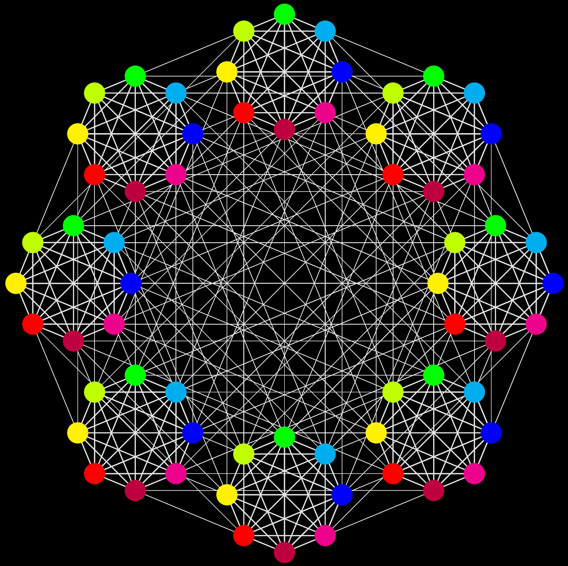
Cartesian Product of Complete Graphs

$$K_n \square K_m$$

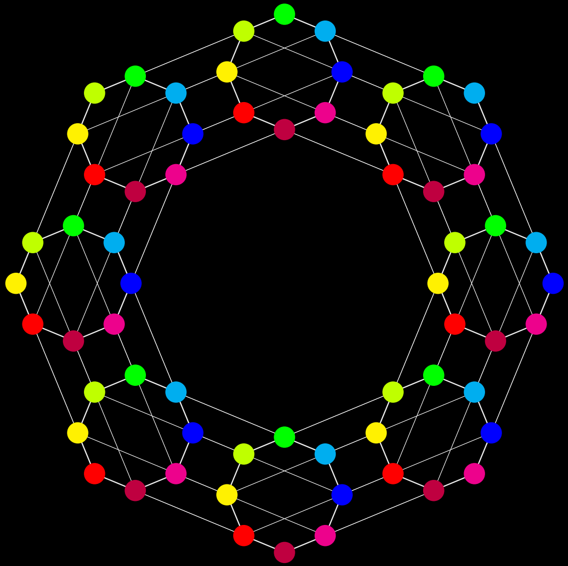
k-Domination of Cartesian Product of Complete Graphs

$$\gamma_k(K_n \square K_m)$$

$K_8 \square K_8$



$C_8 \square C_8$



Cartesian Product of Graphs

If G and H are connected graphs with n and m vertices respectively,

$$\gamma_k(K_n \Box K_m) \leq \gamma_k(G \Box H)$$

The Vizing Conjecture

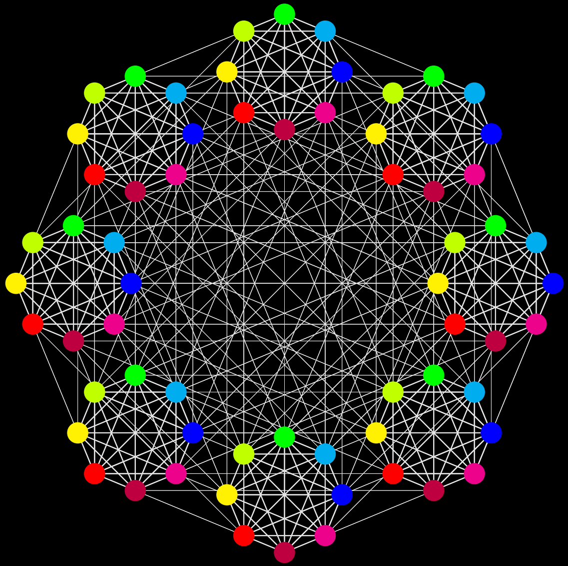
$$\gamma(G)\gamma(H) \leq \gamma(G \Box H)$$

Cartesian Product of Graphs

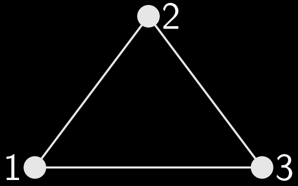
If G and H are connected graphs with n and m vertices respectively,

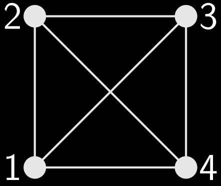
$$\gamma(K_n \Box K_m) \leq \gamma(G \Box H)$$

$K_8 \square K_8$



$K_3 \square K_4$





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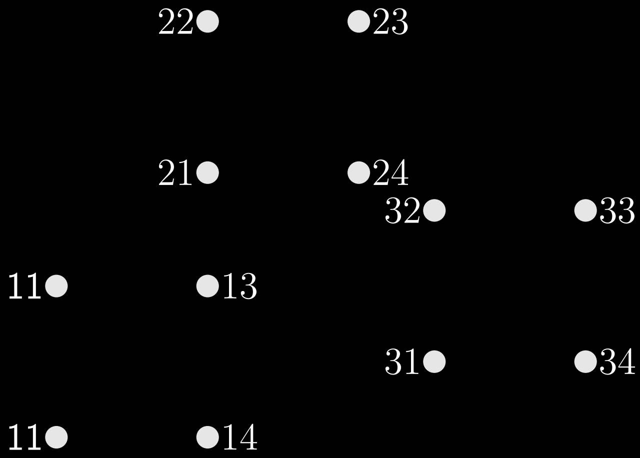

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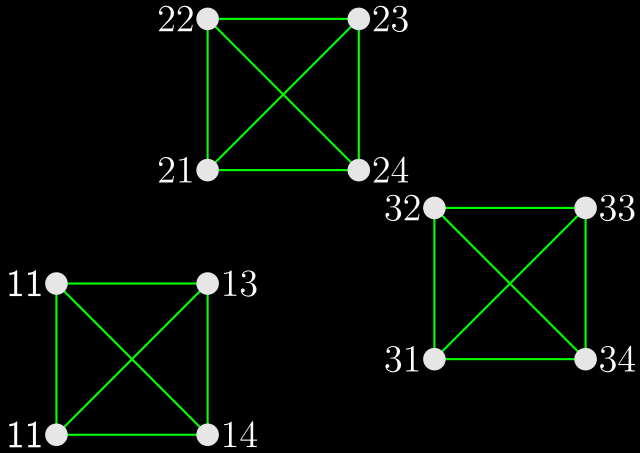

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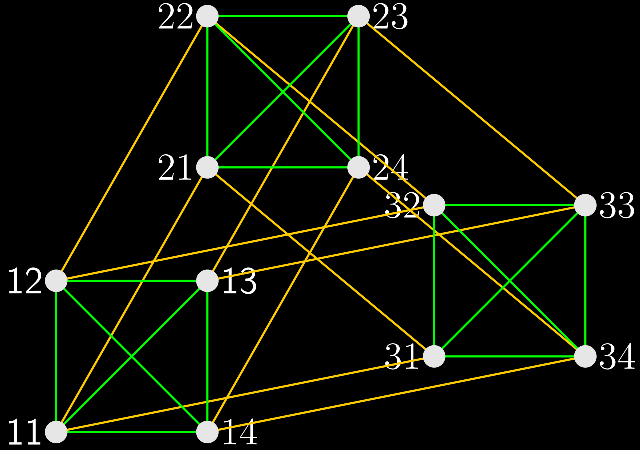
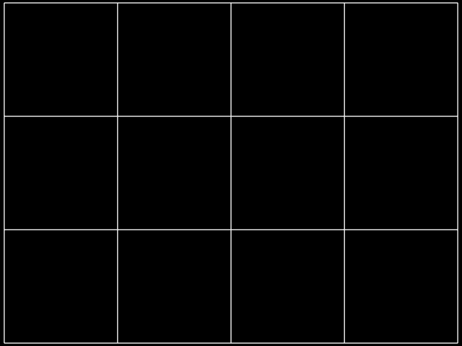


Chessboard: $K_3 \square K_4$



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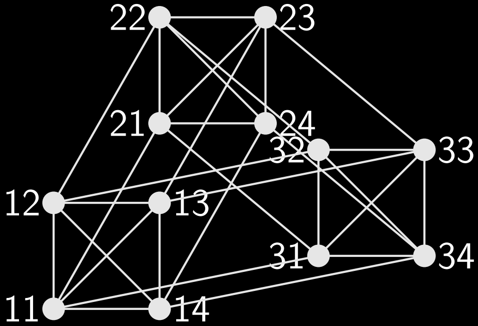

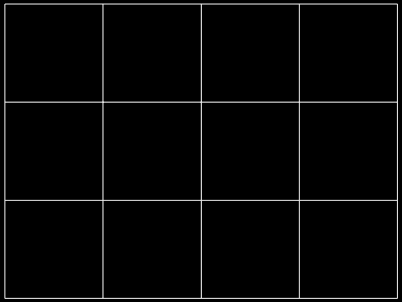

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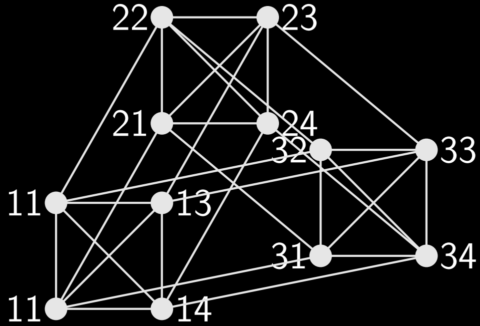
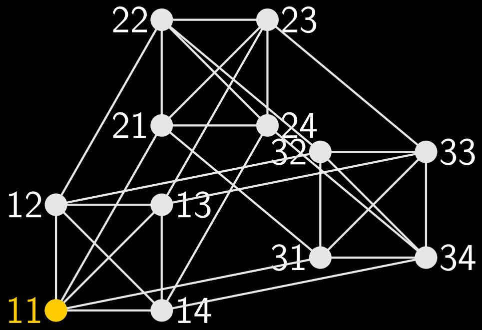


11	12	13	14
21	22	23	24
31	32	33	34

Chessboard: $K_3 \square K_4$



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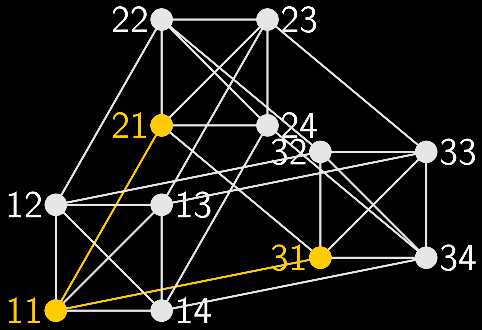

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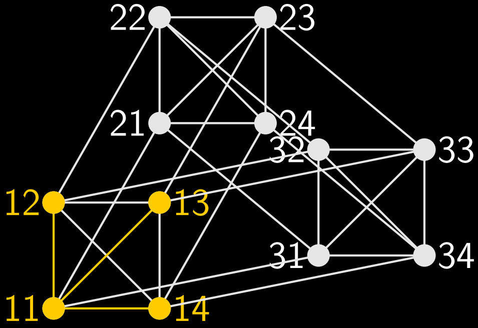

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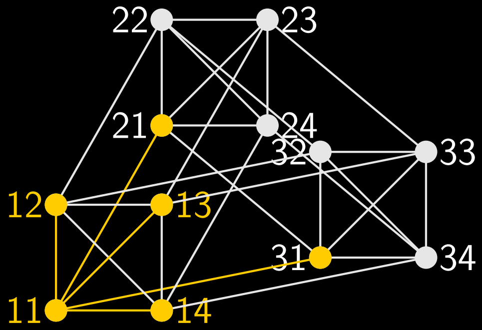

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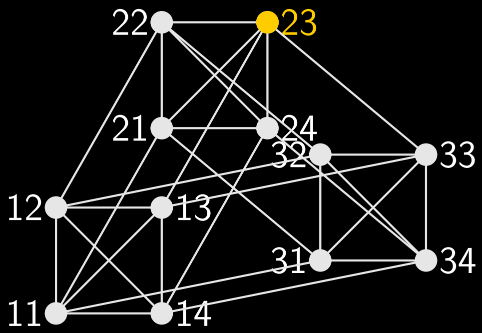

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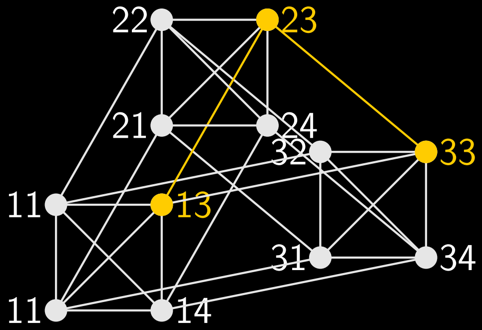

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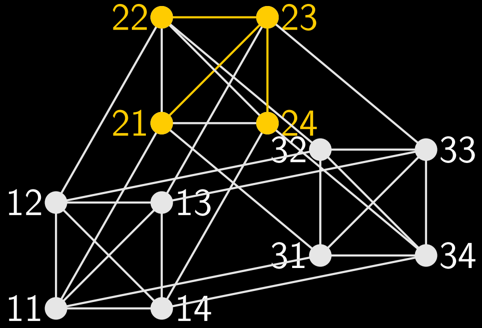
Chessboard: $K_3 \square K_4$



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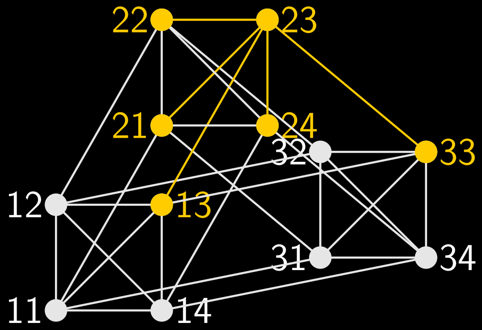

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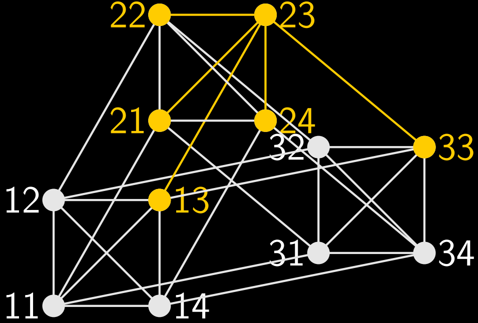

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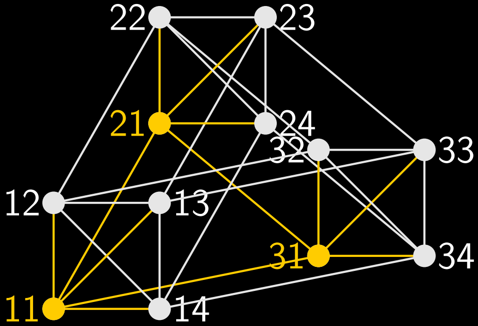

	12	13	14
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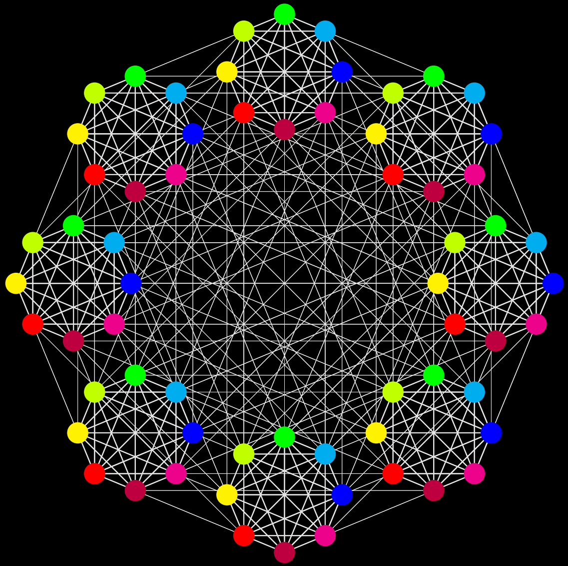
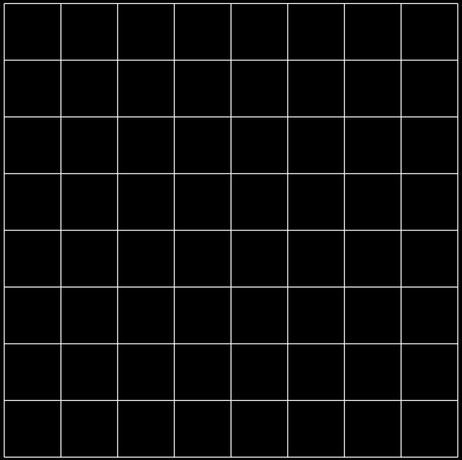
$K_8 \square K_8$



Chessboard: $K_8 \square K_8$



Chessboard: $K_8 \square K_8$

Chessboard: $K_8 \square K_8$

$$\gamma(K_8 \square K_8) = 8$$

Using Chessboards to investigate an Unsolved Conjecture in Graphs

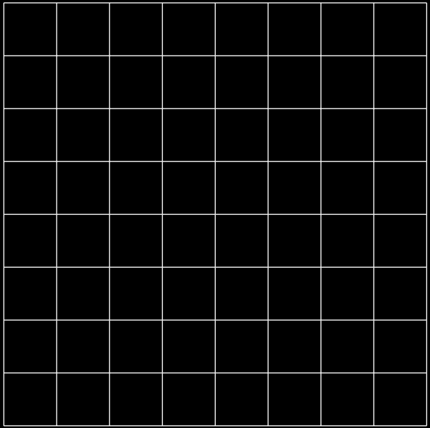
Domination Station

Liam Busch

Rowan University

July 15th, 2022

1-Domination of Square Boards



1-Domination of Square Boards

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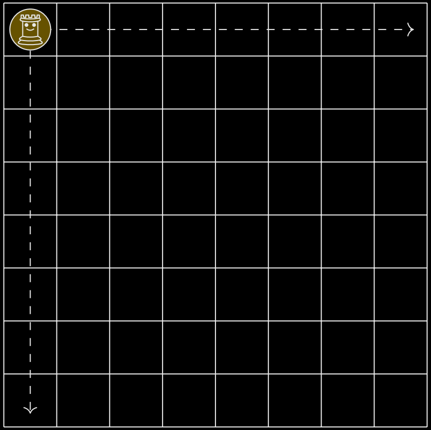
$$\gamma(K_n \Box K_n) \leq n$$

1-Domination of Square Boards



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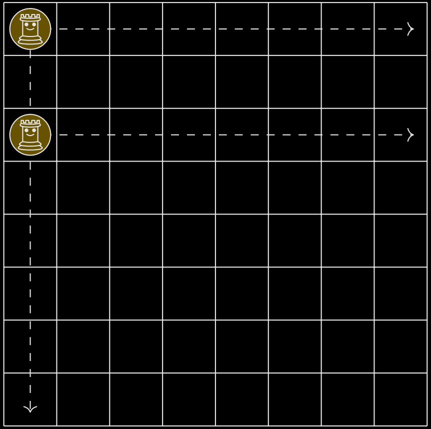

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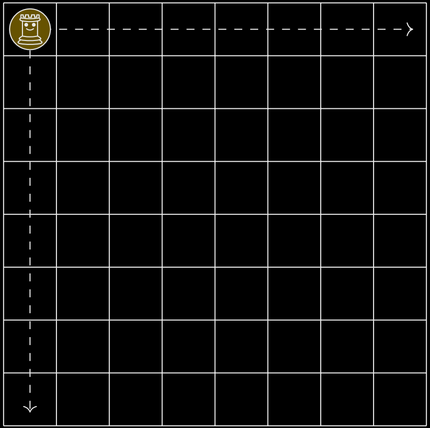

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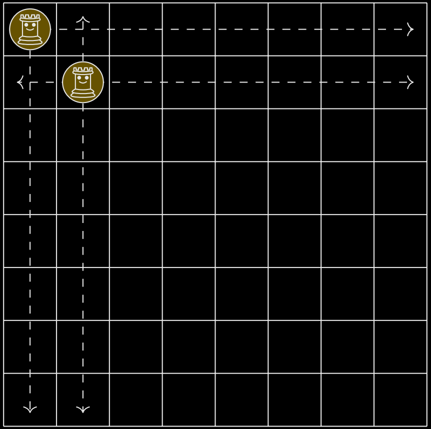

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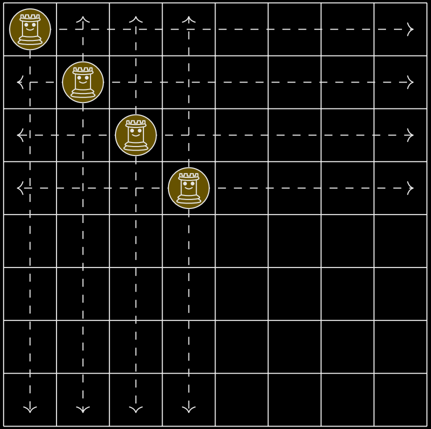

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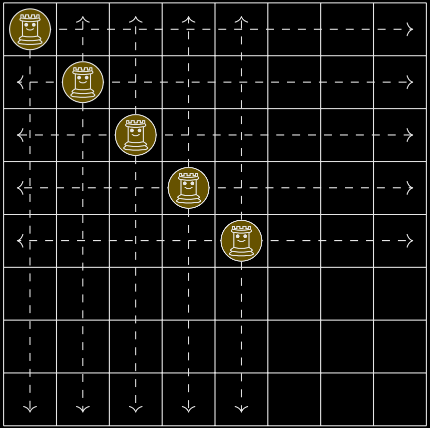

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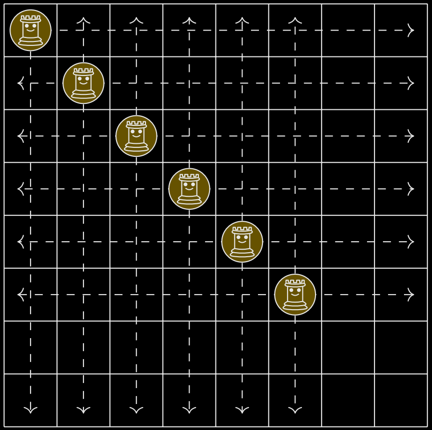

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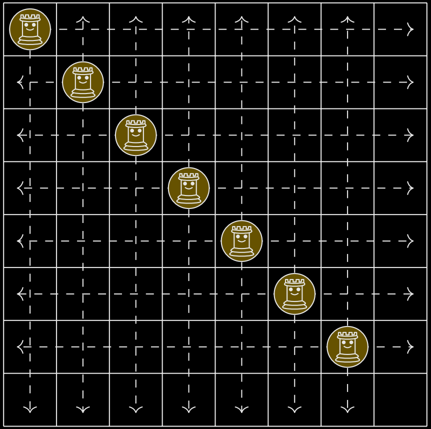

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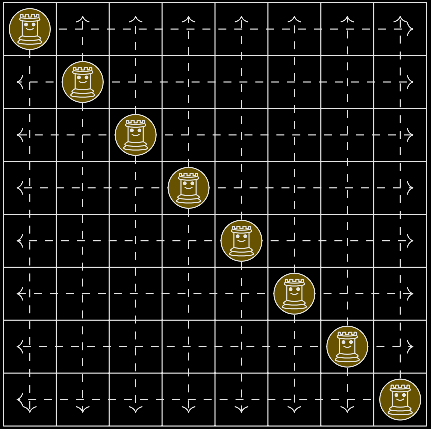

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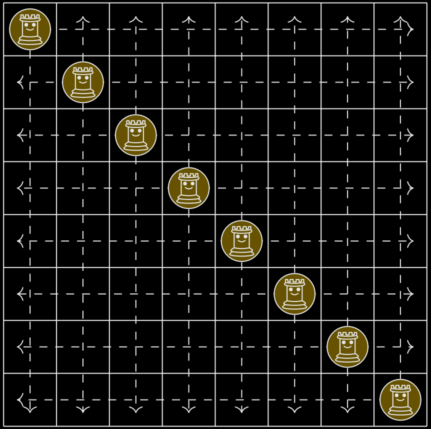


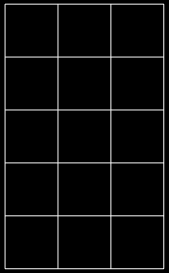
$$\gamma(K_n \square K_n) = n$$

1-Domination of Rectangular Boards

What happens if our board isn't a square?

1-Domination of Rectangular Boards

What happens if our board isn't a square?



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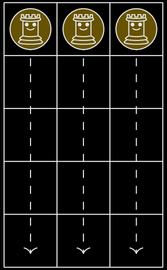

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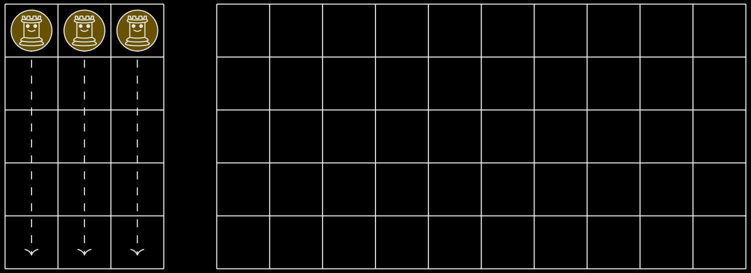

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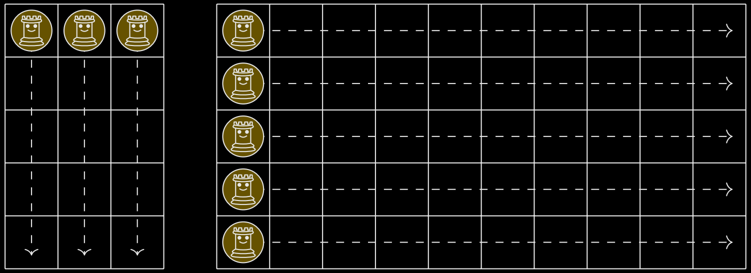

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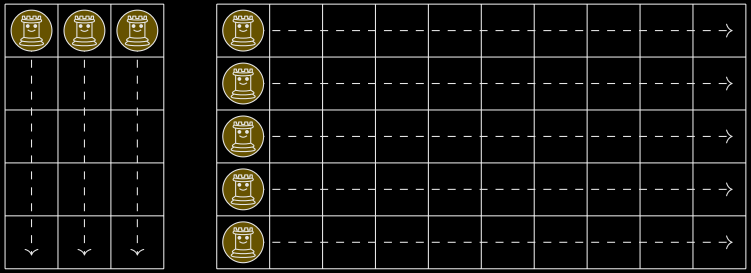

$$\gamma(K_n \square K_m) \leq \min\{n, m\}$$

1-Domination of Rectangular Boards

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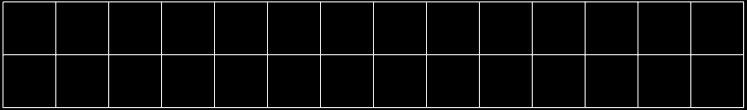

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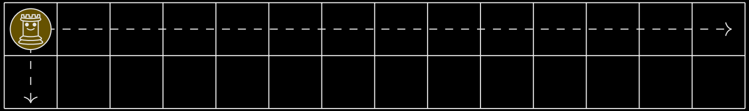

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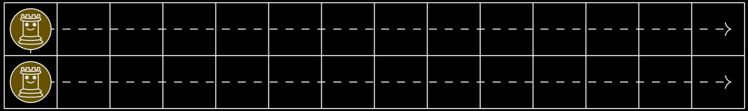

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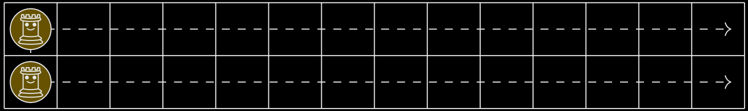

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Symmetry of Cartesian Products

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$K_n \square K_m$ and $K_m \square K_n$ are isomorphic

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$K_n \Box K_m$ and $K_m \Box K_n$ are isomorphic

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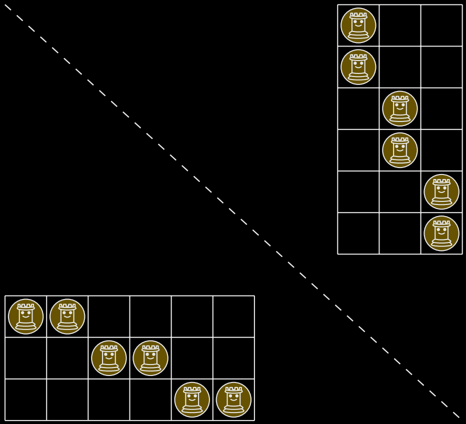

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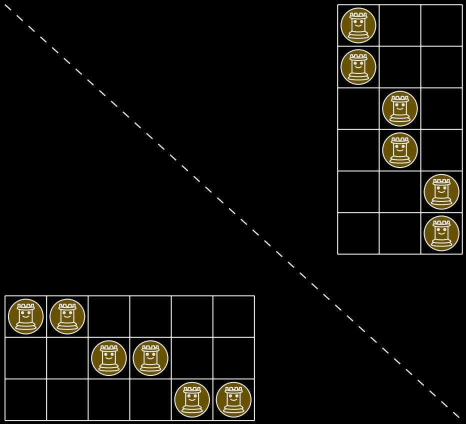

$$\gamma_k(K_n \square K_m) = \gamma_k(K_m \square K_n)$$

2-Domination of Square Boards

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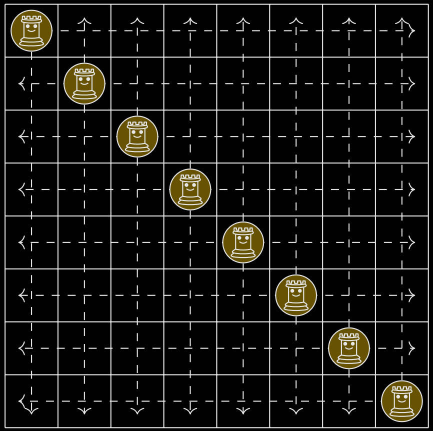

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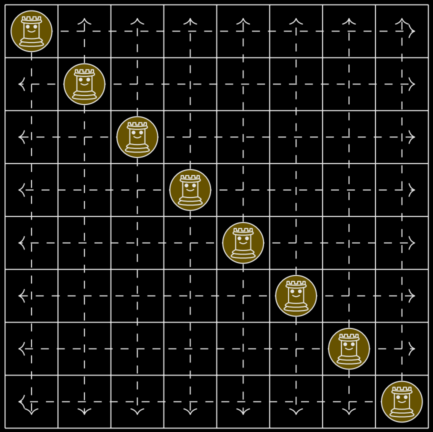

This configuration is 2-dominating!

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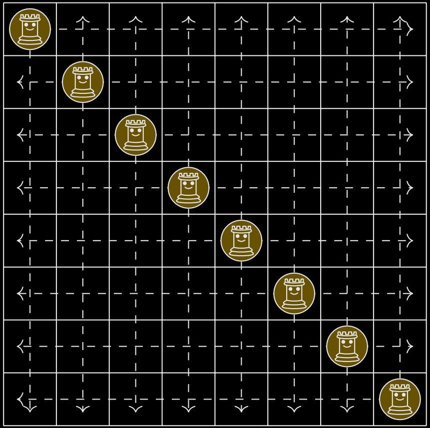

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Is $\gamma_2(K_n \Box K_n) = n$ true?

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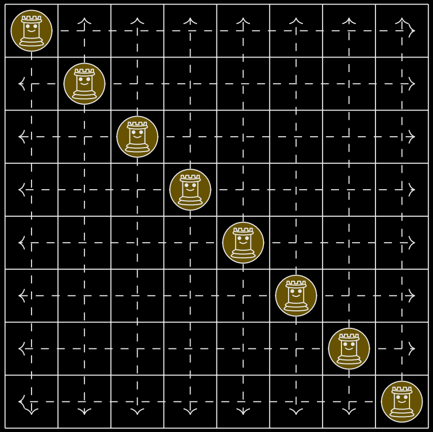

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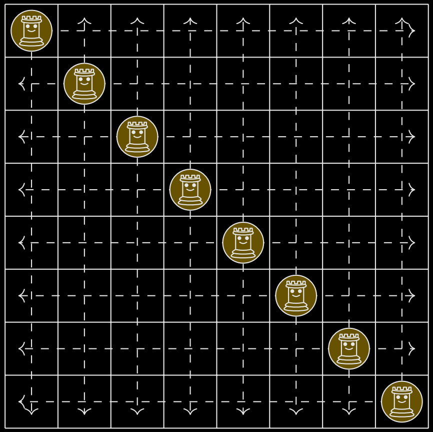

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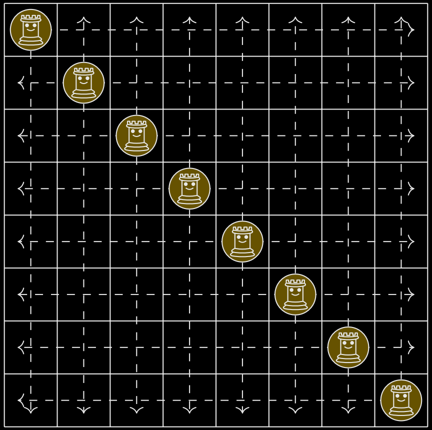
Row/Column Switching



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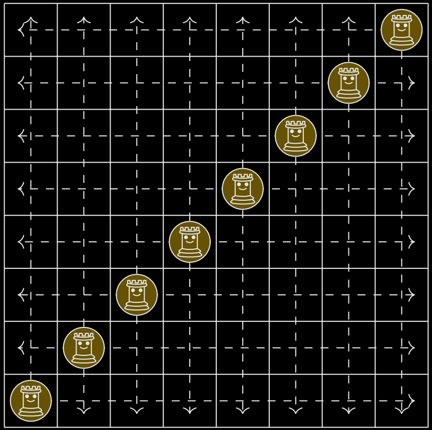

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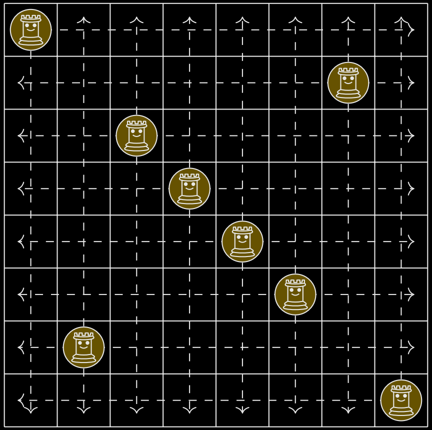

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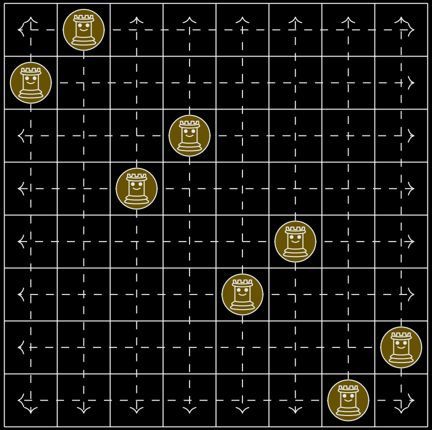

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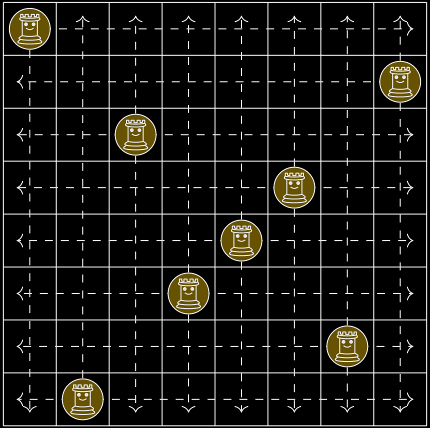

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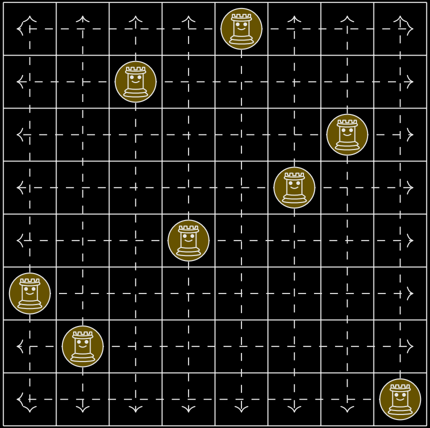

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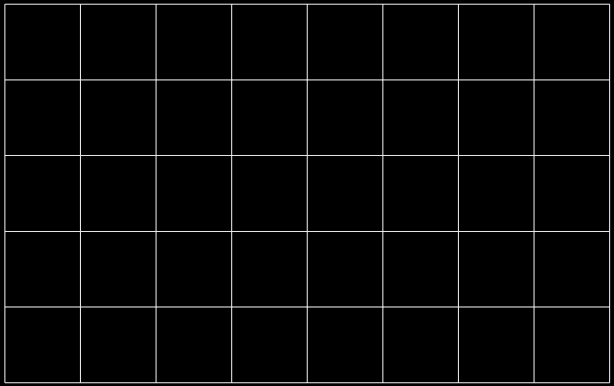

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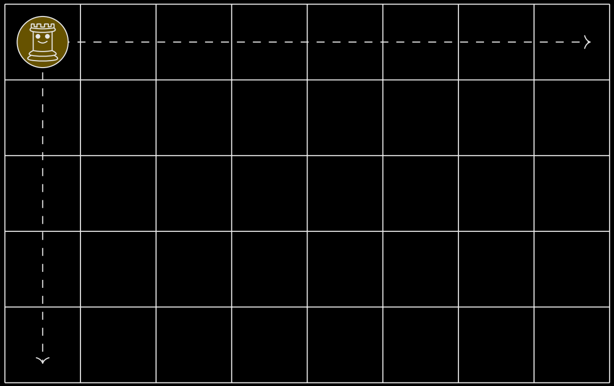

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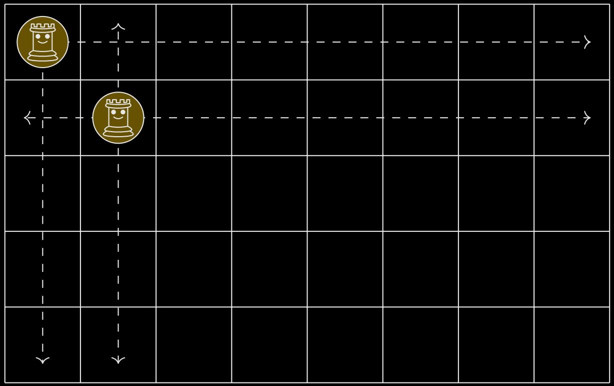

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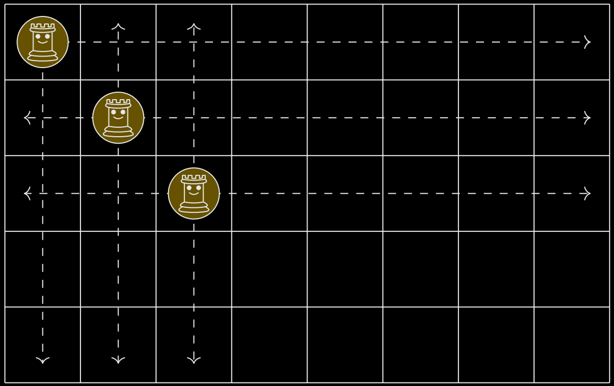

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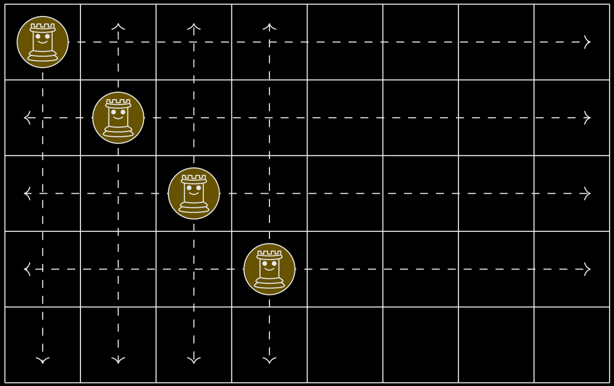

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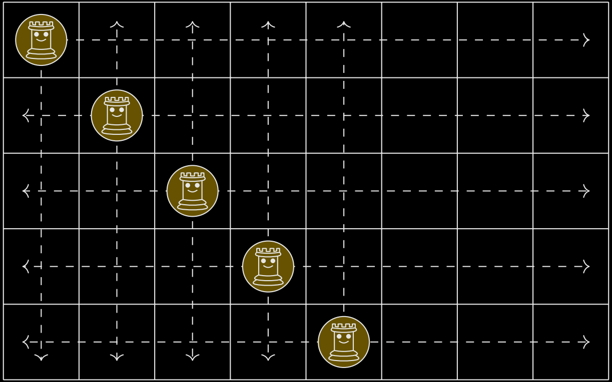

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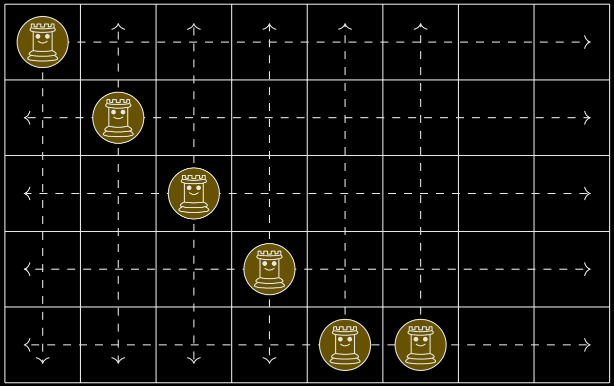

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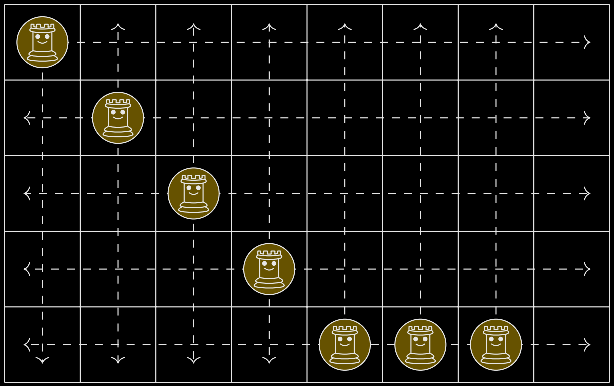

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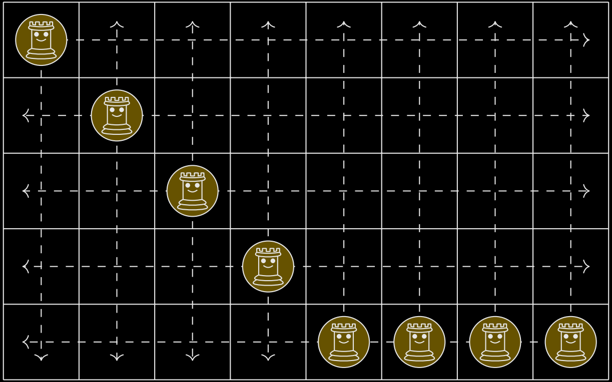

$$\gamma_2(K_n \square K_m) \le \max\{n, m\}$$

The Superposition Principle

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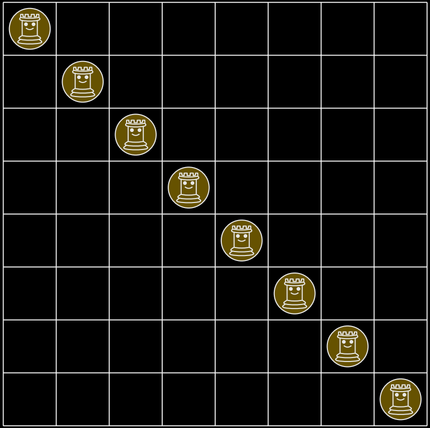
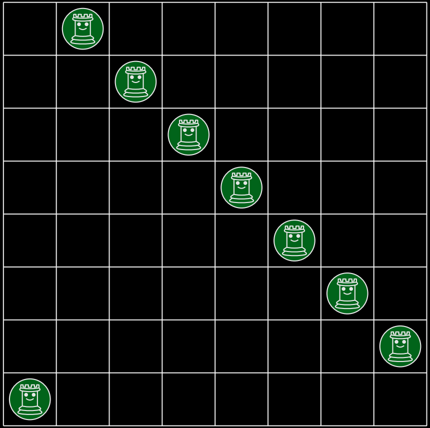
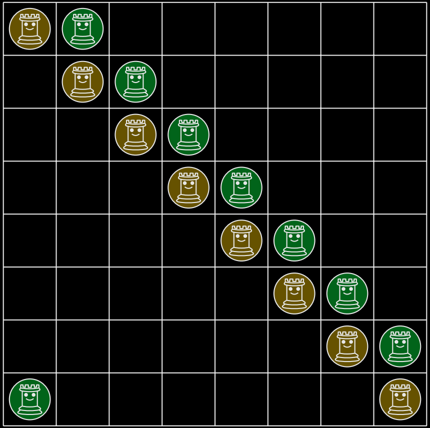




k-dominating

l-dominating

The Superposition Principle



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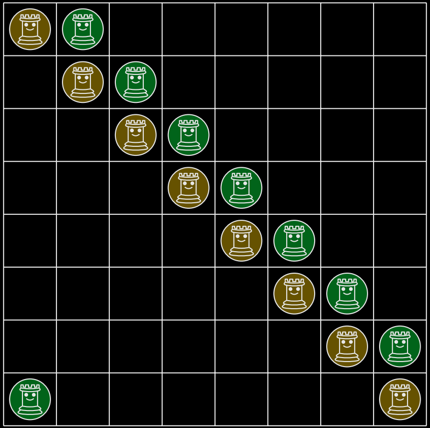


Their union is $(k + \ell)$-dominating!

Using Chessboards to investigate an Unsolved Conjecture in Graphs

Big Square Domination

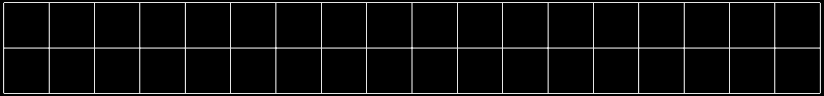
Grant Silewski

University of Minnesota

July 15th, 2022

k-Domination of Rectangles

Large Rectangles ($k = 2$)



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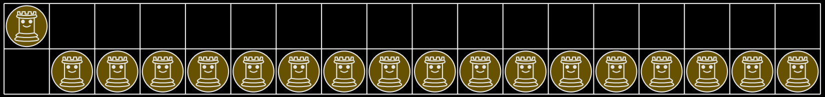

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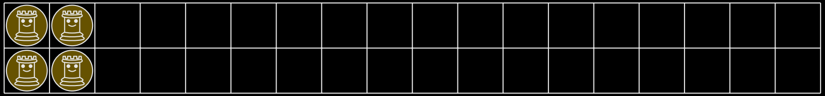
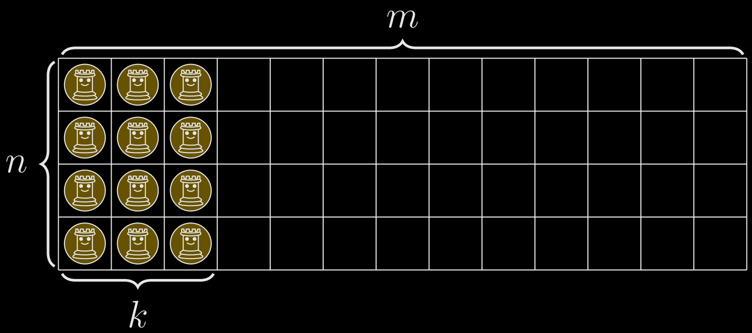


Squareness Argument

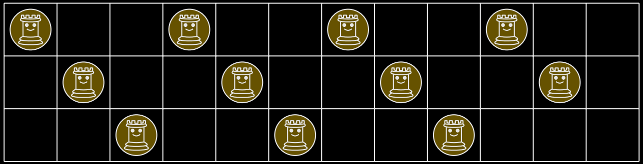
For what m is $\gamma_k(K_n \square K_m) = nk$?



Squareness Argument

$m \geq kn.$

Assume for contradiction $\gamma_k(K_n \square K_m) \leq nk - 1.$

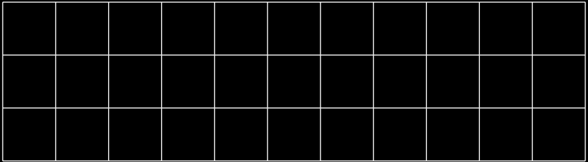


$k = 4$

$\Rightarrow\Leftarrow$

One before squareness Argument

If $n(k-1) \leq m \leq kn$, then $\gamma_k(K_n \Box K_m) = m$.



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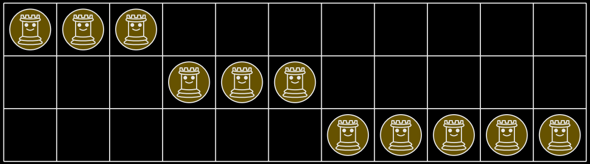
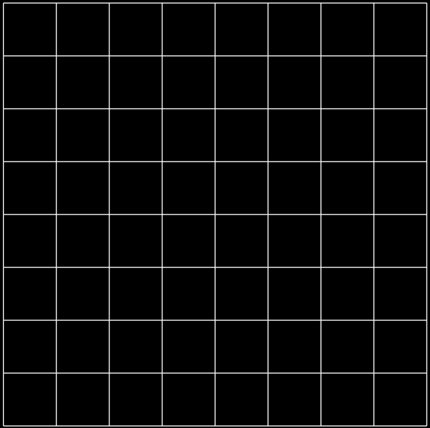


$$k = 4$$

k-Domination of Squares

Upper Bounds of Squares

If $1 \leq k < 2(n-1)$, then $\gamma_k(K_n \square K_n) \leq n \left\lceil \frac{k}{2} \right\rceil$.



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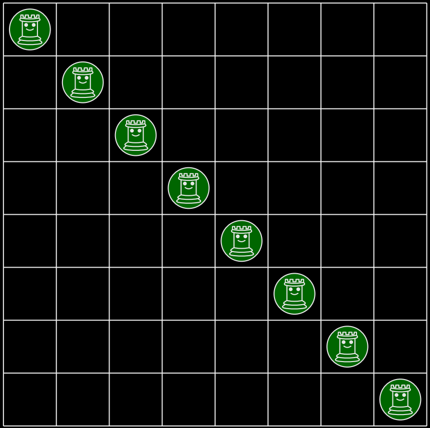

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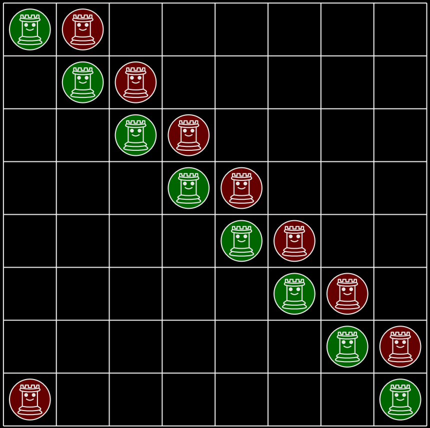

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$k = 7$

Upper Bounds of Squares

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$k = 7$

Proposed Upper Bounds for Squares

For even k

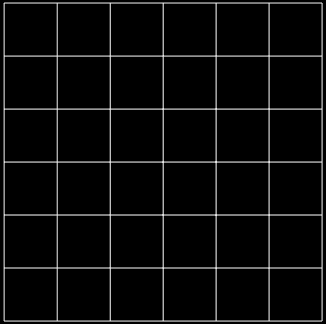
$$\gamma_k(K_n \Box K_n) \leq \frac{nk}{2}.$$

For odd k

$$\gamma_k(K_n \Box K_n) \leq \begin{cases} \frac{(n-1)(k-1)}{2} + n, & for\ k \leq n \\ \frac{(n+1)(k-1)}{2} + 1, & for\ n \leq k < 2(n-1). \end{cases}$$

Proposed Upper Bound for Odd $k \leq n$

$$\gamma_k(K_n \Box K_n) \leq \frac{(n-1)(k-1)}{2} + n$$



$k = 5$

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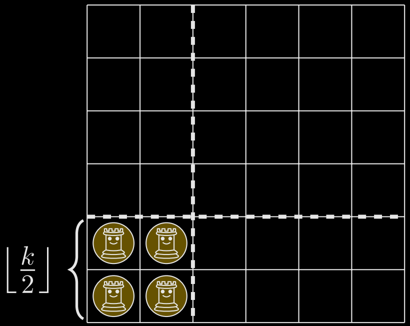

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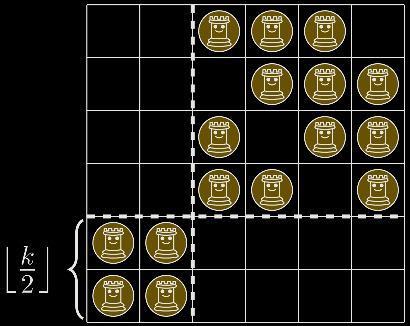

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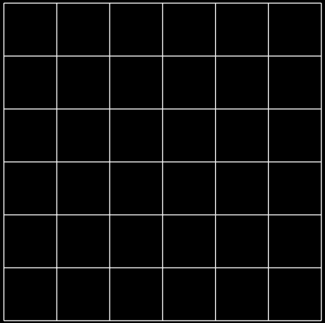
Proposed Upper Bound for Odd k

At some point, can't do a $\left\lfloor \frac{k}{2} \right\rfloor$ square.



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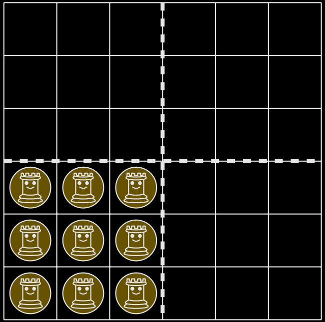
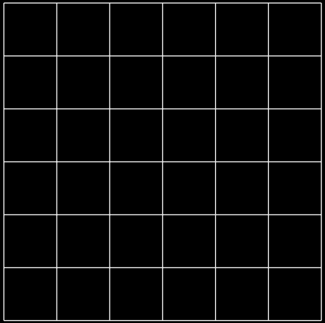


$k = 7$

Proposed Upper Bound for Odd $n < k$

$$\gamma_k(K_n \Box K_n) \leq \frac{(n+1)(k-1)}{2} + 1$$



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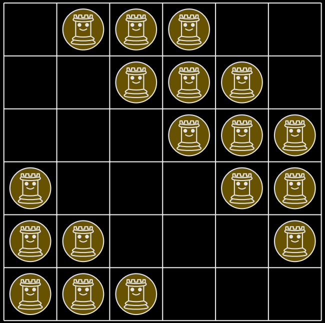

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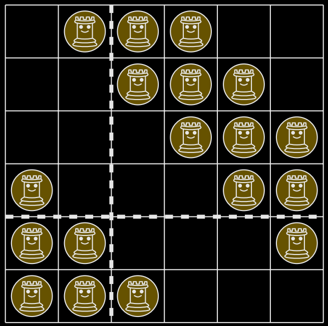

$k = 7$

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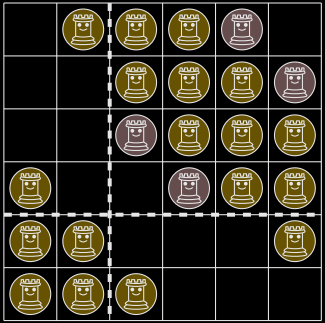


$k = 7$

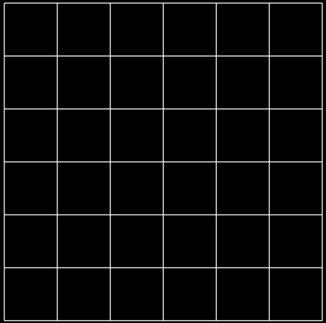
Minimum Rook Lemma for Squares

All rows or columns of an n by n square board contain at least $\left\lfloor \frac{k}{2} \right\rfloor$ rooks.

Minimum Rook Lemma Proof

1. Start with a row containing at most $\left\lfloor \frac{k}{2} \right\rfloor - 1$ rooks.

$k = 8$



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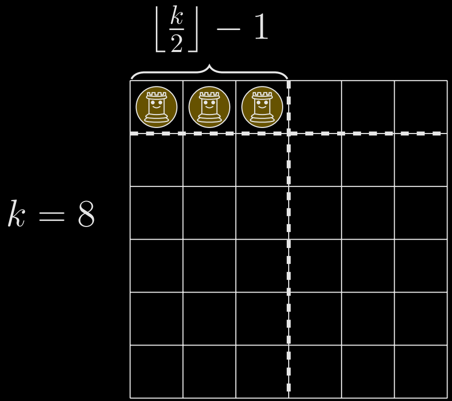
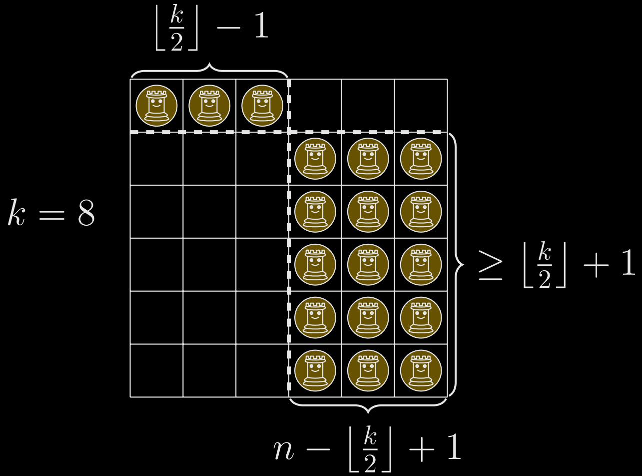


Minimum Rook Lemma Proof

2. Make claims about the remaining rooks.



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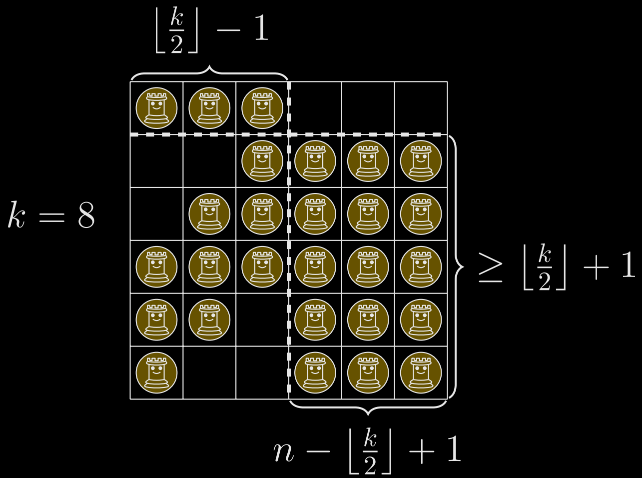


Even k for Squares Closed Formula

For $1 < k \leq 2(n-1)$ and k is even

$$\gamma_k(K_n \Box K_n) = \frac{nk}{2}.$$

Odd k for Squares Closed Formula $k \leq n$

For $1 \leq k \leq n$ and k is odd

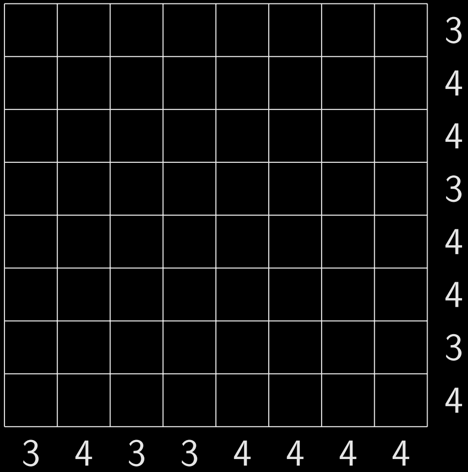
$$\gamma_k(K_n \square K_n) = \frac{(n-1)(k-1)}{2} + n.$$

Odd k for Squares Closed Formula $k \leq n$

Find maximum $\lfloor \frac{k}{2} \rfloor$ rows.

$k = 7$

$n = 8$



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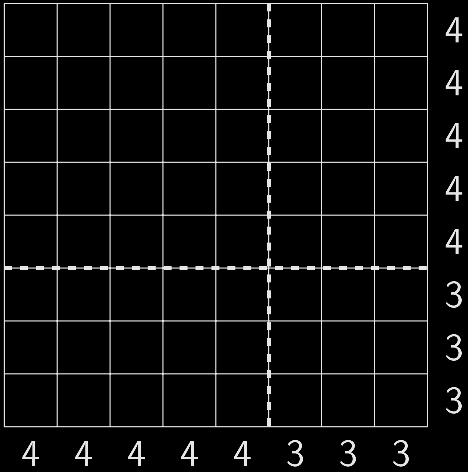

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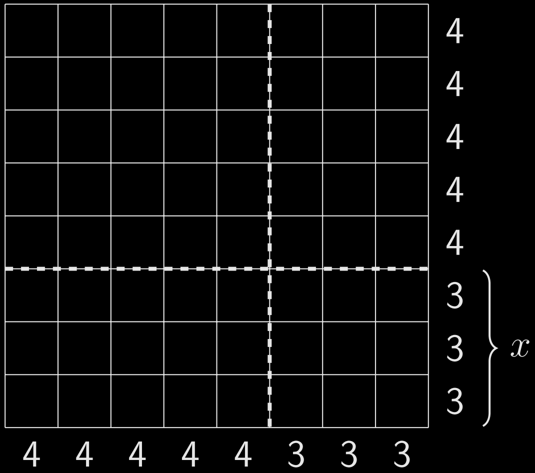

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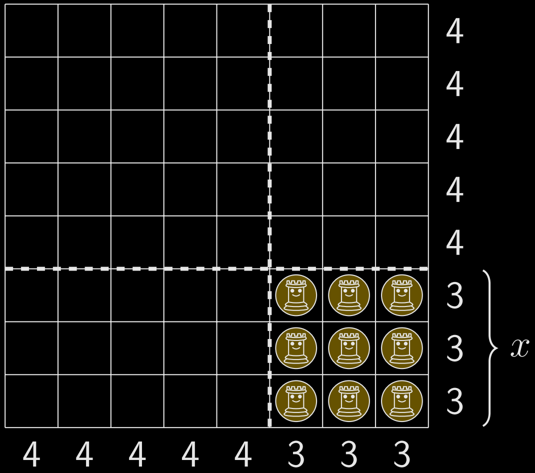

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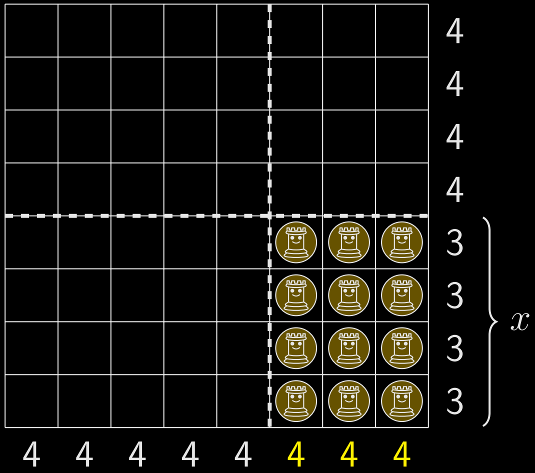

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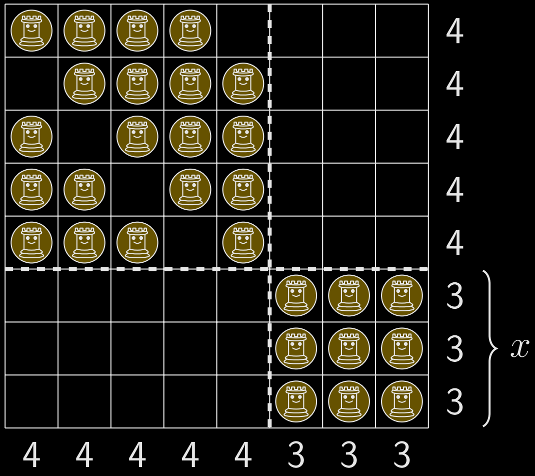

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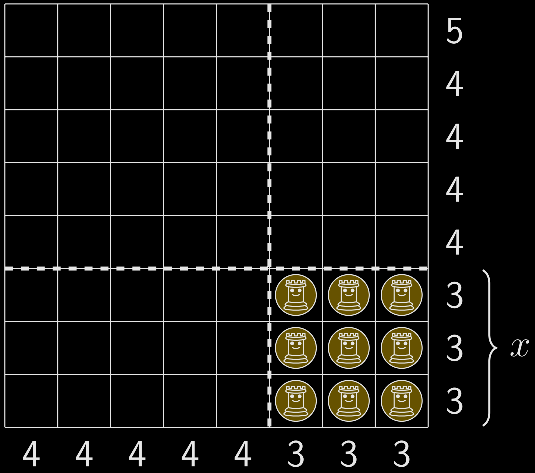

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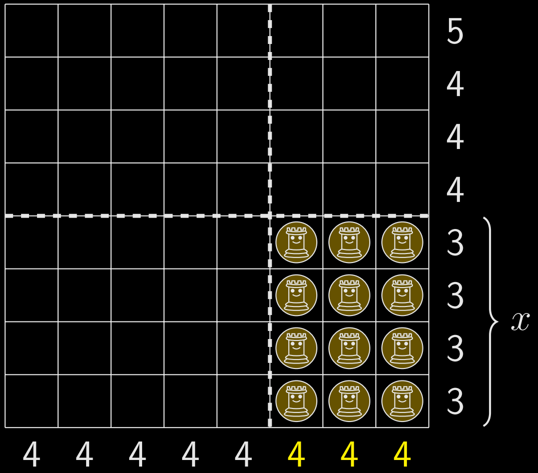

Odd k for Squares Closed Formula $n < k$

For k odd and $n < k \leq 2(n-1)$

$$\gamma_k(K_n \square K_n) = n\left\lceil \frac{k}{2} \right\rceil - \left\lfloor \frac{n}{2}\left(1 - \frac{1}{2n-k}\right) \right\rfloor .$$

Summary and Contributions

1&2-Domination of the Cartesian Product of Complete Graphs

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Summary and Contributions

1&2-Domination of the Cartesian Product of Complete Graphs

Found the cutoff for Trivial Cases

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Minimum Rook Lemma for Squares

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Summary and Contributions

1&2-Domination of the Cartesian Product of Complete Graphs

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Minimum Rook Lemma for Squares

Closed formulas for all Squares of k-domination

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